On magnetohydrodynamic flows with aligned magnetic fields

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The boundary layers due to finite viscosity and magnetic diffusivity are studied in relation to two models of the flow of a conducting fluid past a body in an aligned magnetic field. In each case it is deduced that the growth of the boundary layer may have substantial effects, such as to raise doubts about the validity of the assumed basic flow patterns.

1. Introduction

The calculation of the steady flow of a conducting fluid of small viscosity past a rigid body presents such mathematical difficulties that in most investigations so far attempted it has been assumed that the effect of one or both of viscosity and magnetic diffusivity can be ignored. Until it has been shown that the boundary layers associated with viscosity and diffusivity do not separate or grow in such a way as to disrupt the flow, the results of such investigation must be viewed with suspicion. It is quite insufficient merely to require that the Reynolds number and magnetic Reynolds number shall be large; as for non-conducting flow, this merely implies that viscous and diffusive forces are small in the main body of the fluid. Furthermore, there is no reason why breakdown of the flow should not occur even when there is a corresponding non-conducting flow in which separation is not encountered.

The study of magnetohydrodynamic boundary layers has now reached a stage at which one may hope to make useful deductions about the behaviour of the boundary layers in flows which have been investigated previously with simplifying assumptions. In this paper we examine two such flow situations, each with the magnetic and velocity fields aligned at large distances from the body. In both cases we reach the conclusion that additional boundary-layer effects may be of fundamental importance.

The first situation is the flow past a body in a very strong aligned magnetic field. This has been analysed by Chester (1961) for an arbitrary body of revolution, and by Chester & Moore (1961), in greater detail, for a circular disk normal to the stream. The basic flow pattern they arrive at is that within the cylinder circumscribing the body, with its axis is the stream direction, the fluid is brought to rest, while outside this cylinder the stream is unperturbed. The disturbances to the magnetic field are everywhere small. This is an attractive picture and there seems no reason why it should not apply for a general three-dimensional body as well as for a body of revolution. All the flow and field equations are automatically satisfied inside and outside the cylinder, for a body of the same permeability as the fluid; the only questions is what happens at the interface between the stationary fluid and the stream, where magnetohydrodynamic boundary layers or mixing regions must develop. This type of flow is the magnetohydrodynamic analogue of Kirchhoff free-streamline flow in classical hydrodynamics. With the Alfvén speed greater than the stream speed (as in this problem) there is an upstream wake as well as a downstream one, and the free streamline flow could therefore take this very simple form.

In the present paper an analysis of boundary-layer type is made of the flow near the interface between the stream and the cylinder of stagnant fluid. The leading terms are in exact agreement with Chester's calculation, viscous forces being balanced by the Lorenz forces, while inertial forces and the effects of magnetic diffusivity are of smaller order. However, the next most important terms in the equations are found to have no satisfactory solution. This would seem to indicate that the idea of boundary layers growing in the interface, outwards from the body both upstream and downstream, may not be correct.

An investigation leading to results similar to Chester's has recently been published by Childress (1963). Like Chester, Childress ignores the effects of magnetic-field variations. Although he calculates a second approximation to the flow far from the body, he studies the boundary-layer flow near the body only to the first approximation. Consequently he does not encounter the dilemma met with in this paper.

The second situation we are to examine is the small-disturbance flow past an aerofoil with an aligned magnetic field. This has been studied by Sears & Resler (1959). They ignore both viscosity and magnetic diffusivity, and deduce that the flow pattern is identical with that for a non-conducting fluid. Lary (1960) has considered the effect of magnetic diffusivity so large that the magnetic boundary-layer thickness is large compared with the body thickness or centreline displacement, the fluid remaining inviscid. However, to support Sears & Resler's solution, what is really needed is an assessment of the effect of a boundary layer thin compared with the body thickness, as in ordinary aerofoil theory.

On this basis the validity of the idea of small-disturbance flow has been challenged by Stewartson (1960). He obtains a relation between the changes across a boundary layer in the tangential components of velocity and magnetic field, and uses this to argue that perturbations in the fluid must be large. However, Stewartson's relation applies only if the normal component of magnetic field is not small, and this condition is not satisfied here. Although this appears to discredit Stewartson's criticism it does not confirm Sears & Resler's view, and the conclusion from the study made here is that boundary-layer separation is likely to occur when the field strength is sufficiently great, if the ratio ϵ of the viscous to the magnetic diffusivity is small, as is usually the case in practice. For larger values of ϵ, Sears & Resler's model may well be valid, and in particular for $\epsilon = \infty$ the results of Hasimoto (1959) provide a justification for their analysis for all values of the field strength.

The arguments of Sears & Resler are not dependent on the strength of the magnetic field, and could still apply even when Chester's free-streamline flow is also a possibility. The original motivation of the present investigation was the idea that a study of the boundary layers might enable judgement to be given in favour of one model or the other. In fact the result has been to raise misgivings about both of them. If both flow patterns are to be rejected when the field is strong, the remaining possibility is that the actual solution involves large disturbances to the flow, and wakes of substantial widths extending both upstream and downstream. The immediate prospects for calculating such a flow do not seem bright.

2. The boundary layer at an interface in free-streamline flow

The equations governing steady magnetohydrodynamic flow of a fluid of constant properties are

$$\rho(\mathbf{q}.\nabla)\,\mathbf{q} = -\nabla p + \rho\nu\nabla^2\mathbf{q} + \mu\mathbf{j}\wedge\mathbf{H}, \tag{2.1}$$

$$\mathbf{j} = \nabla\wedge\mathbf{H} = \sigma(\mathbf{E} + \mu\mathbf{q}\wedge\mathbf{H}), \tag{2.2}$$

$$\nabla.\mathbf{q} = \nabla.\mathbf{H} = \nabla\wedge\mathbf{E} = 0, \tag{2.3}$$

in MKS units, where p is the pressure, ρ the density, $\mathbf{q}$ the velocity, $\mathbf{H}$ and $\mathbf{E}$ the magnetic and electric fields, $\mathbf{j}$ the current, ν the kinematic viscosity, σ the conductivity, and μ the permeability.

In Chester's model of the flow in a strong aligned field, as discussed in §1, the boundary layer at the interface between the stream and the stagnant region may be treated as two-dimensional. We measure x downstream from the point of attachment at the body and y normal to it. Then, outside the boundary layer, $\mathbf{q} = U_0\,\mathbf{i}$, $\mathbf{H} = H_0\mathbf{i}$ for $y > 0$, and $\mathbf{q} = 0$ for $y < 0$, where $\mathbf{i}$ is a unit vector in the x-direction.

The analyses of the boundary layers at the interface upstream and downstream of the body are very similar. For the latter, the similarity form of the two-dimensional boundary-layer equations deduced from (2.1), (2.2) and (2.3) is the same as for a flat plate (see Glauert 1961) apart from the boundary conditions. The equations are

$$f''' + ff'' - \beta gg'' = 0, \tag{2.4}$$

$$g'' + \epsilon(fg' - f'g) = 0, \tag{2.5}$$

with boundary conditions

$$f'(-\infty) = 0, \quad f'(\infty) = 2, \quad g'(\infty) = 2. \tag{2.6}$$

[We shall see that these boundary conditions are sufficient.] The independent variable η is given by

$$\eta = \tfrac{1}{2}(U_0/\nu x)^{\frac{1}{2}}\,y, \tag{2.7}$$

and the velocity components (u, v) and magnetic field components (H_x, H_y) are

$$u = \tfrac{1}{2}U_0 f', \quad v = \tfrac{1}{2}(U_0\nu/x)^{\frac{1}{2}}\,(\eta f' - f), \tag{2.8}$$

$$H_x = \tfrac{1}{2}H_0 g', \quad H_y = \tfrac{1}{2}H_0(\nu/U_0 x)^{\frac{1}{2}}\,(\eta g' - g). \tag{2.9}$$

The non-dimensional parameters

$$\epsilon = \sigma\mu\nu, \quad \beta = \mu H_0^2/\rho U_0^2 \tag{2.10}$$

are respectively the ratio of the viscous to the magnetic diffusivity, and the square of the ratio of the Alfvén speed to the stream speed U_0.

We are interested in solutions of these equations when β is large compared with unity. A change of variables is needed so that the last term of (2.4) is of the same order of magnitude as the first term. First, we use (2.5) to write (2.4) in the form

$$f''' + ff'' - \beta\epsilon(f'g^2 - fgg') = 0. \tag{2.11}$$

We now take as independent variable

$$\xi = \beta^{\frac{1}{4}}\epsilon^{\frac{1}{4}}\eta = \tfrac{1}{2}M^{\frac{1}{2}}yx^{-1}, \tag{2.12}$$

and write
$$F(\xi) = \beta^{\frac{1}{4}}\epsilon^{\frac{1}{4}}f(\eta), \quad G(\xi) = \beta^{\frac{1}{4}}\epsilon^{\frac{1}{4}}g(\eta), \tag{2.13}$$

where
$$M = \mu H_0 x(\sigma/\rho\nu)^{\frac{1}{2}} \tag{2.14}$$

is the Hartmann number, based on the distance x. Equations (2.11) and (2.5) become

$$F''' - F'G^2 + FGG' + \beta^{-\frac{1}{2}}\epsilon^{-\frac{1}{2}}FF'' = 0, \tag{2.15}$$

$$G'' + \beta^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}(FG' - F'G) = 0, \tag{2.16}$$

with boundary conditions

$$F'(-\infty) = 0, \quad F'(\infty) = 2, \quad G'(\infty) = 2. \tag{2.17}$$

This formulation will be valid only if the length scale across the boundary layer is small compared with that along it. Hence from (2.12), M must be large, but there is no direct condition on the Reynolds number; this is in full agreement with Chester's findings.

We now suppose that both $\beta^{-\frac{1}{2}}\epsilon^{-\frac{1}{2}}$ and $\beta^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}$ are small. This will be true for sufficiently large values of H_0, for any value of the conductivity parameter ϵ. The conditions may be written in terms of M, the Reynolds number $R = U_0x/\nu$, and the magnetic Reynolds number $R_M = U_0x\sigma\mu$, in the form

$$\beta^{-\frac{1}{2}}\epsilon^{-\frac{1}{2}} = R/M \ll 1, \quad \beta^{-\frac{1}{2}}\epsilon^{\frac{1}{2}} = R_M/M \ll 1, \tag{2.18}$$

precisely as given by Chester. However, while he merely ignored terms of these orders, we shall examine their effects.

We seek solutions of (2.15) and (2.16) in the form

$$F = F_0 + \beta^{-\frac{1}{2}}F_1 + \ldots, \quad G = G_0 + \beta^{-\frac{1}{2}}G_1 + \ldots, \tag{2.19}$$

where F_1, G_1 may depend on ϵ, but not on β. Since ξ does not appear explicitly in the equations, we may without loss of generality impose the additional boundary condition $G(0) = 0$. Then from (2.16) and (2.17), $G_0 = 2\xi$, and (2.15) gives

$$F_0''' - 4\xi^2F_0' + 4\xi F_0 = 0. \tag{2.20}$$

The general solution of (2.20) is

$$F_0 = a\xi + b\left\{e^{-\xi^2} + 2\xi\int e^{-\xi^2}d\xi\right\} + c\left\{e^{\xi^2} - 2\xi\int e^{\xi^2}d\xi\right\}, \tag{2.21}$$

where a, b, c are constants. The solution satisfying (2.17) is

$$F_0 = \pi^{-\frac{1}{2}}\left\{e^{-\xi^2} + 2\xi\int_{-\infty}^{\xi} e^{-t^2}dt\right\}, \quad F_0' = 2\pi^{-\frac{1}{2}}\int_{-\infty}^{\xi} e^{-t^2}dt. \tag{2.22}$$

Equation (2.16) now shows that $G_1'' = -2\pi^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}e^{-\xi^2}$, and hence

$$G_1 = \pi^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}\left\{2\xi\int_{\xi}^{\infty} e^{-t^2}dt - e^{-\xi^2}+1\right\}, \tag{2.23}$$

since the boundary conditions require that $G_1(0) = G_1'(\infty) = 0$. To this approximation the field $H_s\mathbf{i}$ in the stagnant region is given by

$$H_s = \tfrac{1}{2}H_0 G'(-\infty) = H_0(1+\beta^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}). \tag{2.24}$$

Across a magnetohydrodynamic boundary layer $p+\frac{1}{2}\mu H^2$ is constant, and hence the pressure p_s in the downstream stagnant region is given by

$$p_s - p_0 = -\mu H_0^2\beta^{-\frac{1}{2}}\epsilon^{\frac{1}{2}} = -\mu H_0 U_0(\rho\sigma\nu)^{\frac{1}{2}}. \tag{2.25}$$

This again is as found by Chester.

So far the calculation has proceeded smoothly, and all Chester's results have been reproduced exactly, which confirms that we are both dealing with the same physical situation. Only (2.21) contains a hint of trouble to come. The last term, representing the third independent integral of equation (2.10), is totally unacceptable at $\xi = \pm\infty$. This explains why only three boundary conditions can be laid down in (2.17), rather than four as would be anticipated from the orders of equations (2.15) and (2.16), remembering that the zero in ξ is arbitrary. But physically, it reminds us that a magnetohydrodynamic boundary layer with $\beta > 1$ may grow upstream as well as downstream, and here we are arbitrarily insisting that only downstream propagation shall occur. As the boundary layer diffuses there is to be only transmission of vorticity, with no reflexion. Similarly when we come to study the layer upstream of the body, we shall demand that only upstream propagation occurs.

Using the calculated results, the equation for F_1 is

$$\begin{aligned} F_1''' - 4\xi^2F_1' + 4\xi F_1 = {} & 2\pi^{-1}(\epsilon^{\frac{1}{2}}-\epsilon^{-\frac{1}{2}})\left\{e^{-2\xi^2}+2\xi e^{-\xi^2}\int_{-\infty}^{\xi} e^{-t^2}dt\right\} \\ & - 8\pi^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}\xi e^{-\xi^2} - 2\pi^{-1}\epsilon^{\frac{1}{2}}e^{-\xi^2} + 4\pi^{-1}\epsilon^{\frac{1}{2}}\xi\int_{-\infty}^{\xi} e^{-t^2}dt, \end{aligned} \tag{2.26}$$

with boundary conditions $F_1'(-\infty) = F_1'(\infty) = 0$. In view of the complementary functions given in (2.21) we write $F_1 = \xi K$, $K' = \xi^{-2}e^{-\xi^2}L$, and obtain on integrating (2.26) once,

$$\begin{aligned} L' = {} & \pi^{-1}(\epsilon^{\frac{1}{2}}-\epsilon^{-\frac{1}{2}})\left\{3\xi e^{2\xi^2}\int_{-\infty}^{\xi} e^{-3t^2}dt - \xi\int_{-\infty}^{\xi} e^{-t^2}dt\right\} \\ & + 2\pi^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}\xi - 2\pi^{-1}\epsilon^{\frac{1}{2}}\xi e^{\xi^2}\int_{-\infty}^{\xi} e^{-t^2}dt + c_1\xi e^{2\xi^2}, \end{aligned} \tag{2.27}$$

where c_1 is a constant. The term in c_1 integrates to give rise to the unacceptable third complementary function of (2.21). Neither as $\xi\to-\infty$ nor as $\xi\to\infty$ may L' have a term proportional to $\xi e^{2\xi^2}$, or the boundary conditions cannot be met. The condition as $\xi\to-\infty$ requires $c_1 = 0$, and that as $\xi\to\infty$ requires

$$c_1 = (3/\pi)^{\frac{1}{2}}(\epsilon^{-\frac{1}{2}}-\epsilon^{\frac{1}{2}}).$$

Our solution has therefore broken down, except in the single case $\epsilon = 1$. A full solution for this case is given in the Appendix.

For the boundary layer upstream of the body, we measure x upstream from the point of attachment of the interface at the body, so that now $\mathbf{q} = -U_0\mathbf{i}$, $\mathbf{H} = -H_0\mathbf{i}$, for $y > 0$ outside the boundary layer, and consequently in (2.6) and (2.17) $f'(\infty) = g'(\infty) = F'(\infty) = G'(\infty) = -2$ instead of $+2$. The effect of this is to change the sign of F_0 and G_0, but not of F_1 and G_1. In addition $p_s - p_0$ changes sign. This again confirms Chester's result, and implies, from (2.25), that there is a drag force D on the body given by

$$D = 2\mu H_0 U_0 A(\rho\sigma\nu)^{\frac{1}{2}}, \tag{2.28}$$

where A is the projected area of the body in the stream direction. The difficulties associated with the calculation of F_1 remain precisely as before.

How serious is this breakdown of the analysis? In non-conducting fluid mechanics the validity of an inviscid solution can be settled by studying the behaviour of the associated boundary layers. The form of (2.21) and (2.19) strongly suggests that it is the corresponding problem that we are examining here. If so, the results imply that F_0 and G_0 do not give limiting solutions for large values of β, but have the same sort of status that the potential flow past a circular cylinder has in relation to the flow at large but finite Reynolds numbers.

Another possibility is that our analysis has not been sufficiently general, and that the difficulties could be resolved by a more elaborate expansion procedure, as is needed for the low Reynolds number flow past a sphere or a cylinder. It is not easy to see what scheme could succeed here, particularly since the unacceptable contribution to F_1 increases exponentially with ξ, but the possibility cannot be dismissed entirely without further study.

Is it possible to vary the conditions of the problem in such a way that a satisfactory solution exists? It may be verified that the solution still breaks down if we assume that outside the boundary layer $u = U_s \neq 0$ for $y < 0$. In any case this has no obvious application to the flow past a body. A recourse which does succeed is to suppose that the stream U_0 is not constant but varies with x. We assume that outside the downstream boundary layer for $y > 0$,

$$u = U_0(1 + \alpha\beta^{-\frac{1}{2}}\log x), \quad H_x = H_0(1 + \alpha\beta^{-\frac{1}{2}}\log x), \tag{2.29}$$

where α is some constant. [Outside the boundary layer $\mathbf{j} = 0$, and it follows from (2.2) and (2.3) that $\mathbf{q} = (U_0/H_0)\,\mathbf{H}$, since $\mathbf{E} = 0$ in this problem.] We now look for a solution of the boundary-layer equations, derived afresh from (2.1), (2.2) and (2.3), in the form

$$\left.\begin{aligned} u &= \tfrac{1}{2}U_0\{F_0'(\xi) + \beta^{-\frac{1}{2}}\log x\, F_\alpha'(\xi) + \beta^{-\frac{1}{2}}F_1'(\xi) + \ldots\}, \\ H_x &= \tfrac{1}{2}H_0\{G_0'(\xi) + \beta^{-\frac{1}{2}}\log x\, G_\alpha'(\xi) + \beta^{-\frac{1}{2}}G_1'(\xi) + \ldots\}, \end{aligned}\right\} \tag{2.30}$$

where ξ retains its previous definition (2.12). After some calculation it emerges that F_0 and G_0 are unchanged, that

$$F_\alpha = \tfrac{1}{2}\pi^{-\frac{1}{2}}\alpha\left\{e^{-\xi^2} + 4\xi\int_{-\infty}^{\xi} e^{-t^2}\,dt\right\}, \quad G_\alpha = 2\alpha\xi, \tag{2.31}$$

and that the equations for F_1 and G_1 may be satisfactorily solved if

$$\alpha = (3^{\frac{1}{2}}/8\pi)\,(\epsilon^{\frac{1}{2}} - \epsilon^{-\frac{1}{2}}). \tag{2.32}$$

Upstream of the body F_0 and G_0 change sign, but not F_α, G_α, F_1 or G_1. This means that we need a stream velocity and a magnetic field increasing in the flow direction (both ahead of and behind the body) if $\epsilon > 1$, and decreasing if $\epsilon < 1$. The physical implication of this is obscure. Streamwise gradients could of course be produced by confining the flow within a converging or diverging channel, but it is also conceivable that even when unbounded the flow adjusts itself so that this solution becomes applicable near the body.

3. The boundary layer on a slender body with attached flow

We now turn to an examination of the model of the flow suggested by Sears & Resler (1959), in which the ideal fluid flow is the same as that for an inviscid non-conducting fluid, and so remains attached to the body surface. Sears & Resler suppose that such a flow is also meaningful for real fluids, at any rate for thin bodies.

In setting up the corresponding boundary-layer problem, the crucial point to be decided at the start is the question of the conditions to be imposed on the magnetic field at the body surface. For two-dimensional or axisymmetric flow, in order to fix the solution of the boundary-layer equations one condition must be specified relating the normal and tangential components of the field. Likewise to solve the magnetostatic problem for the field within the body, one and only one relation between the components must be specified. Across the surface the tangential component of $\mathbf{H}$ and the normal component of $\mu\mathbf{H}$ are continuous, so there are just the required number of relationships available.

For a non-magnetic body, lines of magnetic intensity which are found within the body must have entered through the boundary layer. In the ideal fluid solution the magnetic field lines are frozen into the fluid and so, like the streamlines, are tangential to the surface, even over the rounded front part of a body. For a real fluid this continues to apply outside the boundary layer. Across the boundary layer the change in the normal component of magnetic field is small (just as the change in the corresponding velocity component is small) and consequently this component is small at the surface. Since magnetic field lines within the body must have entered through the surface, this implies that the field strength within the body is everywhere small, provided that the body thickness is large compared with the boundary-layer thickness, and so the appropriate boundary-layer approximation is that the tangential component of the magnetic field is zero at the surface.

In discussing the boundary layer on a slender body, it is reasonable to take the velocity and magnetic fields outside the boundary layer as constant, equal to their stream values. For convenience of analysis we shall assume that the main part of the body may be represented by a circular cylinder with its axis in the stream direction. If the boundary layer is thin compared with the cylinder radius we then get the magnetohydrodynamic analogue of the classical Blasius boundary layer on a flat plate. In fact the problem is precisely as formulated in equations (2.1) to (2.11) inclusive, if the plate is $y = 0$, $x > 0$, provided that the boundary conditions (2.6) are replaced by

$$f(0) = f'(0) = 0, \quad f'(\infty) = 2, \quad g'(0) = 0, \quad g'(\infty) = 2. \tag{3.1}$$

From (2.9), the condition $g'(0) = 0$ ensures that the tangential component of the field is zero at the surface. In the previous studies of the boundary layer made by Greenspan & Carrier (1959) and Glauert (1961), this condition was replaced by $g(0) = 0$, to give zero normal component of field at the surface. By symmetry this is appropriate for an infinitely thin plate, with a similar boundary layer on each side. In this previous paper the author suggested that the same solution could apply for a thick plate, but he now believes he was mistaken, for the reasons set out above.

For small and large values of the conductivity parameter ϵ, solutions of these equations may be obtained by use of a technique developed in earlier papers (Glauert 1961, 1962). We shall refer to these two papers as **A** and **B**, respectively. [The paper **B** dealt with the case of the magnetic field generated within the body itself, rather than in the external flow.] The key to the method is that when ϵ is either large or small compared with unity, the boundary layer can be treated as two largely separate layers, dominated by viscous and by magnetic forces, respectively, and the solution can be built up in series by matching between the two layers. Solutions starting at $x = 0$ are possible only for $\beta < 1$. The argument given in **A** §6 remains valid, and the result is in accord with the fact that for $\beta > 1$ the wake extends upstream of the body as well as downstream.

The calculation follows so closely those of the earlier papers that there is no need to discuss them at length. For large conductivity, with $\epsilon \gg 1$, the solution in **A** for the boundary condition $g(0) = 0$ has $g'(0) = O(\epsilon^{-\frac{1}{3}})$, and only trivial changes are required to fit the new boundary condition. There seems no reason to suppose that there is any awkward behaviour in the range $0 < \beta < 1$. For small conductivity, with $\epsilon \ll 1$, the previous solution gave $g'(0) = 2 - O(\epsilon^{\frac{1}{2}})$, and here substantial changes are to be expected. In fact the new solutions are surprisingly similar to those in **B** for a magnetized body. For small values of ϵ, the results

$$f''(0) = 1{\cdot}3282\{1 - \tfrac{3}{2}\pi^{-1}\beta \log(1/\epsilon) + O(\beta)\}, \tag{3.2}$$

$$g(0) = 2\pi^{-\frac{1}{2}}\epsilon^{-\frac{1}{2}}\{1 + O(\beta)\}, \tag{3.3}$$

are precisely as in **B** (apart from a sign change in $g(0)$), and suggest, as there, that separation may occur for a value of β of the order of $\{\log(1/\epsilon)\}^{-1}$. [The skin-friction is proportional to $f''(0)$.] The calculation of the separation value is essentially the same as that of **B** §5, but in view of the importance of the result we shall set out the details here. The notation has been slightly simplified, to avoid introducing the mysterious parameter β_0.

We wish to solve equations (2.4) and (2.5) under the boundary conditions (3.1), with ϵ small. In view of the presence of logarithmic terms in the solution for small β it becomes apparent (as in **B**) that the last term in (2.4) is comparable with the second term when it contains a factor $\{\log(1/\epsilon)\}^{-1}$. We define a new parameter

$$X = \beta \log(1/\epsilon), \tag{3.4}$$

and in the inner (viscous) layer write

$$g(\eta) = \epsilon^{-\frac{1}{2}}X^{-\frac{1}{2}}h(\eta). \tag{3.5}$$

Then (2.11) and (2.5) become

$$f''' + ff'' - \{\log(1/\epsilon)\}^{-1}(f'h^2 - fhh') = 0, \tag{3.6}$$

$$h'' + \epsilon(fh' - f'h) = 0. \tag{3.7}$$

Necessary boundary conditions are

$$f(0) = f'(0) = h'(0) = 0. \tag{3.8}$$

In the outer (magnetic) layer the terms of (3.7) are to balance. We write

$$\eta = \epsilon^{-\frac{1}{2}}\zeta, \quad f(\eta) = \epsilon^{-\frac{1}{2}}F(\zeta), \quad h(\eta) = H(\zeta). \tag{3.9}$$

Then $f' = F'$, etc., and the equations become

$$\epsilon F''' + FF'' - \{\log(1/\epsilon)\}^{-1} HH'' = 0, \tag{3.10}$$

$$H'' + FH' - F'H = 0, \tag{3.11}$$

with necessary boundary conditions

$$F'(\infty) = 2, \quad H'(\infty) = 2X^{\frac{1}{2}}. \tag{3.12}$$

Solutions in the inner layer for large η are to match those in the outer layer for small ζ.

We now look for solutions in the form

$$F = F_0 + \{\log(1/\epsilon)\}^{-1} F_1 + \{\log(1/\epsilon)\}^{-2} F_2 + \ldots,$$

etc., where the functions $F_0, F_1, \ldots$ are independent of ϵ but may depend upon X. Equations (3.7), (3.8) and (3.9) indicate that $h_n = \text{const.}$ $H_n'(0) = 0$, and $F_n(0) = 0$, for $n \geqslant 0$. From (3.10), $F_0 = 2\zeta$, and (3.11) then gives

$$H_0'' + 2\zeta H_0' - 2H_0 = 0,$$

of which the required solution is

$$H_0 = 2\pi^{-\frac{1}{2}}X^{\frac{1}{2}}\left\{2\zeta\int_0^\zeta e^{-u^2}\,du + e^{-\zeta^2}\right\}. \tag{3.13}$$

Turning to the inner layer, matching requires that

$$h_0 = H_0(0) = 2\pi^{-\frac{1}{2}}X^{\frac{1}{2}}, \tag{3.14}$$

and from (3.6)

$$f_0''' + f_0 f_0'' = 0. \tag{3.15}$$

This is Blasius's equation, the solution satisfying (3.8) being

$$f_0 = \alpha B(\alpha\eta), \tag{3.16}$$

where $B(\eta)$ is the Blasius function, the solution with $B'(\infty) = 2$, and α is a parameter. Thus $f_0'(\infty) = 2\alpha^2$. We do *not* match f_0 to F_0, since the next terms in the series make contributions of the same order.

The next terms in (3.10) give $2\zeta F_1'' = H_0 H_0''$. As $\zeta \to 0$, $H_0 H_0'' \to 8\pi^{-1}X$ and hence

$$F_1'' \sim 4X/\pi\zeta, \quad F_1' \sim (4X/\pi)\log\zeta + O(1). \tag{3.17}$$

Likewise (3.6) and (3.14) show that

$$f_1''' + f_0 f_1'' + f_0'' f_1 = 4\pi^{-1} X f_0,$$

and hence for η large

$$f_1'' \sim (4X/\pi) f_0'/f_0, \quad f_1' \sim (4X/\pi)\log\eta + O(1)$$
$$= (4X/\pi)\log\zeta + (2X/\pi)\log(1/\epsilon) + O(1). \quad (3.18)$$

Matching of f' and F' demands that

$$2\alpha^2 + \left\{\log\left(\frac{1}{\epsilon}\right)\right\}^{-1}\left\{\frac{4X}{\pi}\log\zeta + \frac{2X}{\pi}\log\left(\frac{1}{\epsilon}\right)\right\} = 2 + \left\{\log\left(\frac{1}{\epsilon}\right)\right\}^{-1}\frac{4X}{\pi}\log\zeta,$$

which requires

$$X = \pi(1-\alpha^2). \quad (3.19)$$

Now $f''(0) = \alpha$, so attached flow with positive skin-friction is possible only for $X < \pi$, or

$$\beta < \{\log(1/\epsilon)\}^{-1}.* \quad (3.20)$$

As the field strength is increased so that this value of β is approached, the skin-friction tends to zero and the boundary-layer separates, as in **B** for precisely the same value of β. The separation is of the same type as occurs on a flat plate in a non-conducting fluid, if there is emission of fluid at the surface with velocity greater than a critical value.

This analysis indicates that when ϵ is small (as it is in most physical applications) attached flow as envisaged by Sears & Resler is possible only for a limited range of magnetic field strengths.

Appendix. Interface solution for $\epsilon = 1$

When $\epsilon = 1$, equation (2.5) becomes

$$g'' + fg' - f'g = 0. \quad (A1)$$

By differentiating this equation and then combining it with (2.4), we obtain the pair of equations

$$h''' + kh'' = 0, \quad (A2)$$

$$k''' + hk'' = 0, \quad (A3)$$

where

$$h = f + \beta^{\frac{1}{2}} g, \quad k = f - \beta^{\frac{1}{2}} g. \quad (A4)$$

The boundary conditions (2.6) show that

$$h'(\infty) = 2 + 2\beta^{\frac{1}{2}}, \quad k'(\infty) = 2 - 2\beta^{\frac{1}{2}}, \quad h'(-\infty) = -k'(-\infty). \quad (A5)$$

For $\beta > 1$, (A5) shows that k is negative for large η, and the only possibility of satisfying (A2) and (A5) is to take $h' = \text{const.} = 2 + 2\beta^{\frac{1}{2}}$. Choosing the zero in η conveniently, we may write $h = 2(1+\beta^{\frac{1}{2}})\eta$, or

$$f = 2(1+\beta^{\frac{1}{2}})\eta - \beta^{\frac{1}{2}} g. \quad (A6)$$

Note that the zero in η used here is not the same as that used in §2. Equation (A1) now becomes

$$g'' + 2(1+\beta^{\frac{1}{2}})\eta g' - 2(1+\beta^{\frac{1}{2}}) g = 0, \quad (A7)$$

* It has been brought to my attention that Stewartson & Wilson (1964) have obtained a similar result recently.

of which the general solution is

$$g = A\eta + B\{2(1+\beta^{\frac{1}{2}})\,\eta\,\mathrm{erf}\,[(1+\beta^{\frac{1}{2}})^{\frac{1}{2}}\,\eta] + \exp\,[-(1+\beta^{\frac{1}{2}})\,\eta^2]\}, \tag{A 8}$$

where A and B are constants. From (2.6) and (A 6) we require

$$g'(\infty) = 2, g'(-\infty) = 2 + 2\beta^{-\frac{1}{2}}.$$

The required solution of the equations is therefore

$$f = \eta\{1 + 2\pi^{-\frac{1}{2}}\,\mathrm{erf}\,[(1+\beta^{\frac{1}{2}})^{\frac{1}{2}}\,\eta]\} + \pi^{-\frac{1}{2}}(1+\beta^{\frac{1}{2}})^{-\frac{1}{2}}\exp\,[-(1+\beta^{\frac{1}{2}})\,\eta^2], \tag{A 9}$$

$$g = 2\eta + \beta^{-\frac{1}{2}}\,\eta\{1 - 2\pi^{-\frac{1}{2}}\,\mathrm{erf}\,[(1+\beta^{\frac{1}{2}})^{\frac{1}{2}}\,\eta]\} - \pi^{-\frac{1}{2}}\,\beta^{-\frac{1}{2}}(1+\beta^{\frac{1}{2}})^{-\frac{1}{2}}\exp\,[-(1+\beta^{\frac{1}{2}})\,\eta^2]. \tag{A 10}$$

It may be verified that the series expansions of these solutions for large β are in accord with the results obtained in §2, allowing for the different zero in η.

In the special case $\epsilon = 1$, solutions of (2.4) and (2.5) have thus been found which are valid for all $\beta > 1$. The arguments of **A** §6 against this possibility were based on the existence of integrals of the equations which grow exponentially for large η. This remains true for (A 3), but the particular boundary conditions of the present problem enable such integrals to be suppressed. For $0 < \beta < 1$, (A 9) and (A 10) continue to be valid solutions, but in this range other solutions of the equations may also exist, with different values of $g'(-\infty)$.

REFERENCES

Chester, W. 1961 *J. Fluid Mech.* **10**, 459.

Chester, W. & Moore, D. W. 1961 *J. Fluid Mech.* **10**, 466.

Childress, S. 1963 *J. Fluid Mech.* **15**, 429.

Glauert, M. B. 1961 *J. Fluid Mech.* **10**, 276 (**A**).

Glauert, M. B. 1962 *J. Fluid Mech.* **12**, 625 (**B**).

Greenspan, H. P. & Carrier, G. F. 1959 *J. Fluid Mech.* **6**, 77.

Hasimoto, H. 1959 *Phys. Fluids*, **2**, 337.

Lary, E. C. 1962 *J. Fluid Mech.* **12**, 209.

Sears, W. R. & Resler, E. L. 1959 *J. Fluid Mech.* **5**, 257.

Stewartson, K. 1960 *J. Fluid Mech.* **8**, 82.

Stewartson, K. & Wilson, D. H. 1964 *J. Fluid Mech.* **18**, 337.